The Transition from Commodity to Fiat Money: an Optimal Seigniorage Approach

Alejandro T. Komai
University of California– Irvine

October 21, 2014

Abstract

This paper uses a search and matching model to elucidate the history of currency. A brief history of currency is provided. The distinction between convertibility and commodity value of money is developed within the context of commitment. Commodity money arose when fiat currency was infeasible. A seigniorage-maximizing model explains the historical trends of decreasing commodity-backing of currency.

Keywords: commodity money, debasement, fiat
The history of money is a history of commodity money. Inconvertible and intrinsically worthless objects maintaining value as a medium of exchange is a fairly recent phenomenon. Previous experiments in suspension of convertibility, as in the United States during the Civil War, or in Britain during the Napoleonic wars, ended with a resumption of convertibility. Experiments in paper money, such as the Law affair in France, and the four big hyperinflations in Europe in the interwar period demonstrate a problem monetary authorities face - the problem of committing not to overprint. To this day, even the US dollar is not without its value as legal tender; it can be used to settle debts public or private.

Considering the social costs involved in using a commodity - such as gold or silver - it is puzzling why we did not arrive in a fiat system sooner. Sovereigns acting as monetary authorities wish to maximize seigniorage - that is, the difference between the price at which they are selling the money good (in real terms) and the fundamental value of the money good. The fundamental value of a fiat money is zero. The fundamental value of a commodity money can be considered the value of the precious metal within the currency.

This paper builds a model to investigate this question. Within the context of a money search environment, the answer has two parts. Fiat money was not always feasible. The feasibility of fiat money depends on three factors: thickness of markets, competitiveness of those markets, and patience of agents. As these three variables change over time - markets grow thicker, competition leads to more trade surplus going to buyers rather than to sellers, changes in life expectancy increase overall patience - the world switches from a regime of commodity money to one of fiat. Secondly, by modeling the choices of a sovereign monetary authority, the paper shows that the seigniorage-maximizing choice for commodity in the money is decreasing in market thickness and decreasing in patience of agents. My model also allows for other comparative statics.

Section 2 provides a review of the literature. Section 3 provides the theoretical environment and an interpretation of how that model fits with the history. Section 4 characterizes steady-state equilibria and performs comparative static experiments. Section 5 concludes. Proofs and figures are in the appendices.

1Shaw (1967), Neal (2000), Sargent and Velde (2002)
2Redish (1993) identifies the collapse of Bretton Woods as the start of the contemporary fiat regime.
3Sargent (1982)
1 History

We live in a world of inconvertible money that is issued by a monopoly, which is also the
government. This money can be used to settle all debts, private or public - it is legal tender.
The world was not always thus.

In the United States, the country’s earliest experiment with paper money was the Con-
tinental, unbacked paper money issued by the Continental Congress. By the end of the
revolution, the Continental was valueless. The Continental Congress faced a problem of
commitment - the commitment not to overprint the currency. This is a perennial monetary
problem. Whether looking at early twenty-first century Zimbabwe, interwar Germany, or
seventeenth century Castile, the problem remains the same. Overprinting the currency leads
to inflation. However, if there are no limits to how much currency the government can print,
then the government cannot stop itself from printing or minting more money in the next
period.

Tying the currency to an intrinsic value was the traditional method to circumvent the
commitment problem. There were two ways to execute this: convertibility and adding
commodity value directly to the currency. Convertibility meant the issuer stood ready to
convert the currency, itself a mere token of value, into its precious metal at face value. Issuers
may be the sovereign or, as in the case of the United States between the Second Bank and
the National Banking System, the issuers may be individual banks. Convertibility had two
effects. The promise to convert gave a value below which the money could not fall. It also
created a restriction on the amount of money that could be issued. Given that holders of
the money may return to the issuer at any point, issuers had to hold sufficient reserves to
satisfy the demands of the money holders.

The other method of commitment is to issue commodity money. In the case of commodity
money issued by the sovereign, the sovereign is limited in how much he can issue based on the
market price of the commodity which will be used in the money. This constraint prevents the
sovereign from overissuing and inflating the money. In the case of free minting, the sovereign
does not choose the amount of currency issued, only the amount of commodity that goes
into the money and the fee charged by the mint for the coinage of new money - this is the
seigniorage fee.

In the model that follows, the seigniorage fee will be the difference between the price
of money and the commodity value of the money. The commodity value of the money will be represented as a presented discounted value of dividends. In the model, the sovereign sells a perpetuity which delivers a dividend. To take this literally requires the belief that the sovereign can commit to pay the dividend. This seems like a strong assumption, but as discussed, this commitment can be seen as the currency being imbued with commodity value at its creation.

2 Literature Review

In Wallace (1980), money is defined as fiat if it is inconvertible and intrinsically useless. Inconvertibility means that there is no one standing by ready to exchange the money for goods or services. Intrinsically useless means money neither delivers direct utility nor can be used for production. It is important to note that the term, “fiat,” originally referred to a money so decreed by the legal authority. In the theoretical sense, the model of the sovereign is only fiat if the dividend on the perpetuity is chosen to be zero. In the historical sense, the money is fiat inasmuch as it is issued by the sovereign - though the sovereign cannot enforce the use of the money and faces the behavior of peasants as a constraint.

In Sargent and Velde (2002) the history of debasements and coin shortages is explained as a process of learning the correct model of managing a monetary system. In their model, convertibility - the promise by government to convert smaller denominations into precious metal - allowed small denominations to circulate within a commodity money regime. The difference between the two is convertibility allows some float for the issuer of the currency and that convertibility requires more commitment from the monetary authority. Commodity money, on the other hand, embeds the commitment within the currency itself. As Redish (1993) describes, the commodity within the currency acts as an anchor to its value.

While this paper does not have something to say directly regarding Gresham’s law, the literature on Gresham’s law does provide a source of inspiration for this research. Fetter (1932) provides a clear history of the origin of Gresham’s law - which was not, in fact, originated by Gresham. While Fetter points to MacLeod and Jevons as the first users of the law, John Stuart Mill’s posthumously published essays, ”On Socialism” seem to echo the law as well. Dutu, Nosal, and Rocheteau (2005) review the literature on Gresham’s

\footnote{An examination of the value to an issuer of float on a note is covered by Wallace and Zhu (2007).}
law - the conjecture that bad money drives out good - as coming from two strains: the legal interventions of exchange rates and asymmetric information. Dutu (2004) provides an excellent anecdote regarding debasements and describes how moneychangers drive out an undervalued currency. Velde, Weber, and Wright (1999) provide a unified framework for discussing Gresham’s law and the history of debasements. They work within the Shi-Trejos-Wright search and matching framework. They provide a theory of ”by tale” and ”by weight” equilibria with a comparative statics interpretation of Gresham. Whereas in the Shi-Trejos-Wright framework it is necessary to model the liquidity gain to agents who turn in their currency during a debasement as an exogenous side payment, the framework in this paper allows an explicit discussion of the liquidity value of money within the model. They find that private information is required to see circulation by tale or Gresham’s law. Rolnick and Weber (1986) contest Gresham’s law and argue that, ”bad money should drive good money to a premium.” Greenfield and Rockoff (1995) provide seven historical cases to test this hypothesis, and find that Gresham’s law survives the tests better than Rolnick and Weber. Redish (1993) provides a history of the transition from commodity money to fiat money and notes, “[...U]nlike the situation in the textbook commodity money world, in practice the monetary authority played a role in the commodity money standard, and the monetary authority could benefit [...] from depreciating the currency.” Sargent and Velde (2002) push this history with respect to a cash-in-advance model for management of commodity money standard. Redish and Weber (2010) provides a search and matching model to micro-found the use of money without resorting to cash-in-advance.

In Ritter (1995), a stylized history is presented with a monetary search model to explain the puzzle of why, if it were optimal from the perspective of a seigniorage-maximizing government to print fiat money, do we not observe fiat money everywhere in history. The strength of this paper is in the simplicity of the model to explain this puzzle. Namely, Ritter’s resolution to the puzzle is that a government which cannot credibly commit itself not to overprint its currency will only be able to produce intrinsically worthless currency in cases where the public knows the government, constrained by its own impatience and size, is seigniorage-maximizing by not overprinting. Another strength of the Ritter model is its examination of the transition path from barter to fiat currency.

It is unfortunate Ritter’s stylized history does not match actual history. The history

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of money begins with credit, not barter, as has been documented by anthropologists and archaeologists of Mesopotamia. Money as we model it arises exactly when the frictions which make money essential encroach upon credit arrangements - public records become unwieldy, villages and cities become too large for reputation to be common knowledge. When money does arise it does not replace barter, it augments credit; it is not fiat money, it is commodity money. The history of currency in Europe according to Shaw (1896) is one of continuous debasement. Concurrently, trade across Europe expanded; markets became more competitive and integrated.

3 Environment

The environment is similar to Lagos and Wright (2005) and follows Nosal and Rocheteau (2011). The economy consists of a continuum of agents called buyers and a continuum of agents called sellers, each normalized to unity. Time is discrete and continues forever. Each period of time is divided into two stages. In the first stage, agents trade consumption goods in a decentralized market (DM) where they are matched bilaterally and at random. In the second stage, agents can trade assets and goods in a competitive market (CM). The good traded in the CM is taken as the numéraire. DM and CM goods perish between stages. During the CM both are capable of production and consumption. During the DM the buyers wish to consume but cannot produce and sellers do not wish to consume but can produce. In addition to these agents, collectively referred to as peasants, there is a single agent called the sovereign, which cannot produce or consume in the DM but can produce and consume in the CM.

The lifetime expected utility of a buyer is given by

$$E \sum_{t=0}^{\infty} \beta^t [u(q_t) + x_t],$$

where \( \beta = 1/(1 + r) \in (0, 1) \) is a discount factor, \( q_t \in \mathbb{R}_+ \) is the consumption of the DM good, \( x_t \in \mathbb{R} \) is the consumption of the numéraire good; when \( x_t < 0 \) it is interpreted as production. The utility function in the DM, \( u(q) \), is defined on \([0, \infty)\), is increasing, twice differentiable, \( u'(0) = \infty, u'(\infty) = 0, u'(q) > 0, u''(q) < 0, \) and \( u(0) = 0 \).
The lifetime expected utility of a seller is
\[ \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[ -c(q_t) + x_t \right], \]
where \( c \) is the DM production technology, defined on \([0, \infty)\), is increasing, twice differentiable, \( c'(0) = 0, c'(\infty) = \infty, c'(q) > 0, c''(q) > 0\), and \( c(0) = 0 \).

Agents in the DM lack commitment and individual trading histories are private information. As there exists a continuum of agents, agents who have matched in the DM will meet again with probability zero. Buyers that match with sellers in the DM cannot commit to repay sellers at a future date because the threat of punishment is, in expectation, zero. Unsecured credit cannot be used; money is essential.

The sovereign’s preferences are given by
\[ \sum_{t=1}^{\infty} \beta^t x_t, \]
where \( x_t \) is the consumption (or production, if negative) in the CM. The sovereign sells a fixed supply, \( A \), of perpetuities at the beginning of time in the CM at market price \( \phi \) in terms of the numéraire. Perpetuities are recognizable (cannot be counterfeited), durable, divisible, and portable (costless to carry). The dividend promised on the perpetuities, \( \kappa \in \mathbb{R}_+ \), is chosen by the sovereign and paid to the bearer of the asset in the CM each period. Assume the sovereign can commit to its promise to pay \( \kappa \) each period.

3.1 Interpretations

On the face of it, the assumption that a sovereign can commit to pay dividends may seem strong. This assumption can be reinterpreted as the sovereign choosing the precious metal value of currency. The literal commodity aspect of the currency can be interpreted as delivering \( \kappa \) in three different ways: (i) the commodity is accepted by foreign merchants so there is some outside option to the money, as in Velde, Weber, and Wright (1999), (ii) the precious metals are intrinsically valued - people enjoy holding shiny objects, (iii) the money can be melted down at any point if the precious metal in the coin becomes more valuable than the circulating value. In this sense, the present discounted value of all future dividend
payments are paid out by the sovereign when the currency is minted.

4 Equilibrium

Begin by studying the steady-state, which assumes the numéraire price of money is constant over time.

4.1 Peasants

Equilibria are characterized by moving backward from the description of peasants’ choices in the CM to the determination of quantities in the DM.

Let \( W^b(a) \) denote the lifetime expected utility of a buyer in the CM holding \( a \) perpetuities in units of the numéraire good. Let \( V^b \) be its value function in the DM. This gives

\[
W^b(a) = \max_{x,a' \geq 0} \left[ x + \beta V^b(a') \right]
\]

\[ \text{s.t. } x + a' \phi = a(\phi + \kappa). \] (2)

The first term in (1) is the consumption (or production if negative) of CM good. The second is the discounted continuation value in the next period. The household chooses CM consumption/production, and liquid assets, \( a \), to maximize lifetime expected utility subject to the budget constraint (2). The left side of the budget constraint is composed of CM consumption/production and purchase of perpetuities, \( a' \). Gross returns to assets are on the right side of the budget constraint. This can be rewritten more succinctly as

\[
W^b(a) = a(\phi + \kappa) + \max_{a' \geq 0}[-\phi a' + \beta V^b(a')].
\] (3)

Note that \( W^b(a) \) is linear in its wealth, \( a(\phi + \kappa) \). The choice of assets for the next period, \( a' \), is independent of the assets held at the beginning of the period, \( a \).

The value function for a buyer holding \( a \) real units in the DM is

\[
V^b(a) = \sigma \{ u(q) + W^b[a - d(q)] \} + (1 - \sigma)W^b(a).
\] (4)

With probability \( \sigma \) buyers match with sellers and trade \( d(q) \leq a \) for \( q \) units of DM good.
The linearity of $W^b(a)$ allows this to be rewritten as

$$V^b(a) = \sigma[u(q) - d(q)(\phi + \kappa)] + W^b(a).$$

(5)

Terms of trade in the DM are determined through a proportional bargaining rule, where the buyer receives a constant share, $\theta$, of the total surplus, and the seller receives the remaining $(1 - \theta)$\(^6\). The buyer faces the following problem:

$$\max_{q \geq 0} \left[ u(q) - d(\phi + \kappa) \right]$$

(6)

subject to

$$u(q) - d(\phi + \kappa) = \frac{\theta}{1 - \theta} [d(\phi + \kappa) - c(q)]$$

(7)

and

$$d \leq a.$$  

(8)

The buyer maximizes utility from consumption of DM good net of a transfer of assets, $d$ to the seller. The value to the buyer of the assets $d$ is $a(\phi + \kappa)$. The assets a buyer can transfer are limited to those brought into the DM, (8). The trade surplus is divided according to (7), which pins down $d$. This can be rewritten as

$$\max_{q \geq 0} \theta[u(q) - c(q)]$$

(9)

subject to

$$\theta c(q) + (1 - \theta)u(q) \leq a(\phi + \kappa).$$

(10)

Note that (10) will hold at equality if the buyer does not bring more than enough assets to afford $q^\ast$. For ease of discussion, define $z(q) \equiv \theta c(q) + (1 - \theta)u(q)$. From the properties of $u$ and $c$, $z(0) = 0$, $z(\infty) = \infty$, $z'(q) = \theta c'(q) + (1 - \theta)u'(q) > 0$, $z'(0) = \infty$, and $z'(\infty) = \infty$. As $z$ is increasing, $z$ is invertible. Concavity of $z(q)$ is determined by its second derivative. See figure 1. If the buyer brings assets into the DM such that $a(\phi + \kappa) \geq z(q^*),$ then the buyer will offer $d = z(q^*)/(\phi + \kappa)$ in exchange for $q = q^*$ from the seller. Otherwise, if $a(\phi + \kappa) < z(q^*)$, the buyer will offer $d = a$ in exchange for $q = z^{-1}[a(\phi + \kappa)]$.

Advancing (5) one period ahead to substitute $V^b(a)$ into the maximization problem in

\(^6\)As discussed in Aruoba, Rocheteau, and Waller (2007), proportional bargaining differs from Nash bargaining in its axiom of strong monotonicity, which has significant implications for the efficiency of trade equilibria. Proportional bargaining does not exhibit the inefficiently low amounts traded Nash bargaining can prescribe.
and applying the proportional bargaining solution yields

$$\max_{a \geq 0} \left\{ -\phi a + \beta \{ \sigma \theta [u(q) - c(q)] + W^b(a) \} \right\}$$

Linearity of $W^b$ and independence of future choices of asset holdings allow this problem to be rewritten as

$$\max_{a \geq 0} \left\{ -r(\phi - \phi^*) a + \sigma \theta \{ u[q(a)] - c[q(a)] \} \right\}, \quad (11)$$

where $\phi^* = \kappa/r$, the fundamental value of the perpetuity. The solution to this problem is given by

$$-r(\phi - \phi^*) + \sigma \theta \mathcal{L}(q)(\phi + \kappa) \leq 0, \quad (12)$$

where $\mathcal{L}(q) \equiv [u'(q) - c'(q)]/[\theta c'(q) + (1 - \theta) u'(q)]$. Note that $\mathcal{L}(0) = 1/(1 - \theta)$, $\mathcal{L}(q^*) = 0$, and $\mathcal{L}'(q) < 0$. Figure 2 illustrates $\mathcal{L}(q)$. This figure depicts $\mathcal{L}$ as concave, which is guaranteed

Figure 1: The amount of money a buyer trades to a seller in exchange for DM goods, $z(q) \equiv \theta c(q) + (1 - \theta) u(q)$.
with a technical assumption on the relative third derivatives of $u(q)$ and $c(q)$.

**Assumption 4.1.** For all $q \in [0, q^*]$, $c'''(q) > 0$ and

$$u'''(q) - c'''(q) < 2c''(q) \frac{u''(q)c'(q) - c''(q)u'(q)}{[c'(q)]^3}.$$  

**Lemma 4.2.** If Assumption 4.1 is true, then $L(q)$ is concave.

![Diagram](image)

**Figure 2:** The liquidity term, $L(q) \equiv [u'(q) - c'(q)]/[\theta c'(q) + (1 - \theta)u'(q)]$.

The corner solution, $q = 0$, occurs when

$$-r \frac{\phi - \phi^*}{\phi + \kappa} + \frac{\sigma \theta}{1 - \theta} < 0. \quad (13)$$

When $\kappa = 0$, this provides a condition on the price such that fiat money will be infeasible. Figure 3 illustrates the region in $\theta - \sigma$ space where fiat money is feasible and where it is infeasible.

When the solution is interior, there are three cases to consider. If $\phi - \phi^* < 0$, then (11) has no solution; unbounded $a$ will be demanded. If $\phi - \phi^* = 0$, then any $a \geq z(q^*)/(\phi + \kappa)$...
is a solution. If $\phi - \phi^* > 0$, then demand will be single-valued\footnote{To prove this, note that when \eqref{12} has an interior solution,}

where $a = z(q)/(\phi + \kappa)$. As $\phi - \phi^* > 0$, this implies $u'(q) - c'(q) > 0$ where $q = z^{-1}[a(\phi + \kappa)]$, and hence $q < q^*$.

Sellers have no demand for assets. Let $W^s(a)$ denote the lifetime expected utility of a seller entering the CM with $a$ perpetuities. Let $V^s$ be the value function for a seller in the DM. This gives

$$W^s(a) = \max_{x,a' \geq 0} \left[ x + \beta V^s(a') \right]$$

s.t. $x + a'\phi = a(\phi + \kappa)$. \hspace{1cm} (14)

$$r \frac{\phi - \phi^*}{\phi + \kappa} = \sigma \theta L(q).$$

The left-hand side is increasing in $\phi$. The right-hand side is decreasing in $\phi$ and $a$. 

\footnote{To prove this, note that when \eqref{12} has an interior solution,}
This can be rewritten as

\[
W^s(a) = a(\phi + \kappa) + \max_{a' \geq 0}[-a'\phi + \beta V^s(a')].
\]

Note that, as with the buyer’s CM value function, the seller’s CM value function is linear in asset holdings and its choice of assets next period is independent of previous holdings.

The value function for the seller holding \(a\) units of assets in the DM is

\[
V^s(a) = \sigma\{W^s(a + d) - c(q)\} + (1 - \sigma)W^s(a).
\]  

(16)

This can be rewritten as

\[
V^s(a) = \sigma[d(\phi + \kappa) - c(q)] + W^s(a)
\]

using the linearity of \(W^s\). Applying the solution to the proportional bargaining problem and substituting \(V^s\) into \(W^s\):

\[
\max_{a \geq 0} \{-a\phi + \beta\sigma(1 - \theta)[u(q) - c(q)] + a(\phi + \kappa)\}.
\]

Unlike the buyer, for whom a change in the amount of assets brought into the DM determines the amount of DM good traded, the amount of DM good the seller receives does not depend on the assets the seller carries. The problem, then, reduces to

\[
\max_{a \geq 0}[-a(\phi - \phi^*)].
\]

There are three cases to consider. If \(\phi - \phi^* < 0\), then there is no solution; unbounded \(a\) will be demanded. If \(\phi - \phi^* = 0\), then any \(a\) is a solution. As the seller is indifferent, one can assume zero assets are demanded. If \(\phi - \phi^* > 0\), then demand will be zero.

Aggregate demand is the correspondence

\[
A^d(\phi) = \left\{ \int_{[0,1]} a(i)\,di : a(i) \text{ is a solution to (11)} \right\},
\]  

(17)

where \([0,1]\) is the measure of buyers and \(a(i)\) is the asset demand for buyer \(i \in [0,1]\).
Aggregate demand, \( A^d(\phi) \) is the correspondence \( [z(q^*)/(\phi + \kappa), \infty) \) for \( \phi = \phi^* \) and equals \( z(q)/(\phi + \kappa) \) for \( \phi > \phi^* \).

To summarize, the buyer takes \((\phi, \kappa, A)\) as given and chooses \((q, d, a)\). If \( \phi > \phi^* \), these decisions are described by

\[
q = \begin{cases} 
q^* : & a(\phi + \kappa) \geq z(q^*) \\
\frac{z(q)}{\phi + \kappa} : & \text{else}
\end{cases}
\]  \hspace{1cm} (18)

\[
d = \frac{z(q)}{\phi + \kappa}
\]  \hspace{1cm} (19)

\[
r(\phi - \phi^*) = \sigma \theta (\phi + \kappa) L(q)
\]  \hspace{1cm} (20)

where \( z(q) \equiv \theta c(q) + (1 - \theta) u(q) \) and \( L(q) \equiv [u'(q) - c'(q)]/[\theta c'(q) + (1 - \theta) u'(q)] \).

### 4.2 Sovereign

The sovereign faces the following program:

\[
\max_{\lambda \geq 0, \kappa \geq 0, \phi \geq 0} A (\phi - \phi^*)
\]  \hspace{1cm} (21)

\[
s.t. \ A \in A^d(\phi).
\]  \hspace{1cm} (22)

The price of money is determined endogenously according to the pricing equation \([12]\). The dividend is restricted to being positive. A negative dividend would require the sovereign to have the power to enforce collection of the negative dividend each CM. While a commodity money interpretation of the model might cast this as money that is costly to carry, a sovereign free to dispose of a costly asset would do so immediately and prefer to employ a fiat money.

The sovereign’s program can be rewritten as

\[
\max_{\kappa \geq 0, \phi \geq 0} \theta z(q) L(q)
\]  \hspace{1cm} (23)

The following technical assumption on the relative third derivatives of the functions \( u(q) \) and \( c(q) \) guarantees the maximand is concave.
Figure 4: The sovereign’s problem can be rewritten to maximize $\theta z(q)L(q)$.

Assumption 4.3. For all $q \in [0, q^*]$, $c'''(q) > 0$ and

$$u'''(q) - c'''(q) < 2c''(q)\frac{u''(q)c'(q) - c''(q)u'(q)}{[c'(q)]^3} - \frac{c''(q)[u'(q)]^4}{c(q)[c'(q)]^4}.$$

Lemma 4.4. If Assumption 4.3 is true, then Assumption 4.1 is true.

Lemma 4.5. If Assumption 4.3 is true, then $z(q)L(q)$ is concave.

Taking the first order condition, for finite $\kappa$,

$$\frac{\mathcal{L}'(q)/\mathcal{L}(q)}{z'(q)/z(q)} = -1. \quad (24)$$

The $\mathcal{L}(\cdot)$ function captures a liquidity term, as can be seen in (20). The $z(\cdot)$ function captures the cost to the buyer in a bilateral trade, as in (19). Equation (24) says that the sovereign will choose $\kappa$ such that the buyer’s trade cost elasticity of the liquidity term is unit. The
Figure 5: The demand $A^d$ for assets and supply $A$ chosen by the sovereign to maximize seigniorage, the shaded region.

intuition behind this is simple. A buyer demands assets to trade more of the DM good. As the access to assets rises, so does the trade cost in a match. This is maximized at $q^*$. However, at $q^*$ there will be no liquidity premium. The liquidity premium is decreasing in $q$. The sovereign trades off the demand for assets, which is increasing in $q$, and the captured liquidity premium, which decreases in $q$. This is understood most clearly by noting that, in equilibrium, $q = \mathcal{L}^{-1}\{[r(\phi - \phi^*)]/[\sigma \theta (\phi + \kappa)]\}$, which is increasing in $\kappa$.

Let $q$ denote the quantity of DM good the buyer chooses given the sovereign’s choice of $(\kappa, A, \phi)$. From (20):

$$q = \mathcal{L}^{-1}\left[\frac{r(\phi - \phi^*)}{\sigma \theta (\phi + \kappa)}\right].$$

As $\mathcal{L}(q)$ is decreasing in $q$, $\mathcal{L}(0) = 1/(1 - \theta)$, $\mathcal{L}(q^*) = 0$, and the fact that $r > 0$, $\phi$ is a decreasing function of $q$, see right graph of Figure 5. At $q^*$, there is no liquidity premium, so the asset is priced at its fundamental value, $\phi^*$. If the price were to fall below the fundamental value, $q^*$ would still be achieved, though an infinite quantity of the asset would be demanded. The price of the asset is increasing in the fundamental value.
4.3 Steady-State Equilibria

Comparative statics are examined with steady-state equilibria.

**Definition 1.** A steady-state equilibrium is a sequence of \( \{q_t, \phi_t, d_t, a_t, A_t, \kappa_t\}_{t=0}^{\infty} \) such that

1. \( q_t = q, \phi_t = \phi, d_t = d, a_t = a, A_t = A, \kappa_t = \kappa \) \( \forall t \geq 0 \),

2. agents bargain to determine \( \{q, d\} \) (6), (7), (8),

3. agents choose \( a \) to maximize utility (11),

4. the sovereign chooses \( \{A, \kappa\} \) to maximize seigniorage subject to the demand for assets (21), (22),

5. prices follow \( z(q) \equiv \theta c(q) + (1 - \theta) u(q) = a(\phi + \kappa) \),

6. and markets clear \( A = a \).

**Proposition 4.6.** In a steady-state equilibrium, fiat currency \((\kappa = 0)\) is feasible when

\[
-r + \frac{\sigma \theta}{1 - \theta} \geq 0. \tag{25}
\]

**Corollary 4.7.** In a steady-state equilibrium,

(i) Fiat currency will be infeasible for any \((\sigma, \theta) \in [0, 1] \times [0, 1)\) pair if \( r \to \infty \).

(ii) Fiat currency will be feasible for any \((\sigma, \theta) \in [0, 1] \times [0, 1]\) pair if \( r = 0 \).

The curve in Figure 3 graphs \( \theta = r/(r + \sigma) \), the boundary of the corner solution to (12). To the left, the graph approaches but does not attain \((0, 1)\). To the right, the graph attains \((1, r/(1 + r))\). The region, Monetary equilibria with fiat, is where, for a given \( r \), the parameters \( \sigma \) and \( \theta \) are such that fiat money \((\kappa = 0)\) is feasible. Below this region, only commodity money can circulate; this region attains the corner solution to (12) for \( \kappa = 0 \). Note that as \( r \to \infty \) fiat money becomes infeasible for all \((\sigma, \theta)\) pairs. As \( r \to 0 \), the right part of the graph approaches \((1, 0)\) and Monetary equilibria with fiat becomes the entire
region $[0, 1] \times [0, 1]$. The interpretation of Proposition 4.6 and Figure 3 is as markets grow thicker ($\sigma$ increases), or as the markup decreases ($\theta$ increases), or as peasants become more patient ($r$ decreases), fiat currency can become feasible. This means that commodity money can be feasible for some values of ($\sigma, \theta, r$), where, for those same values, fiat is infeasible. Without a commodity value to the currency peasants are unwilling to circulate money.

**Proposition 4.8.** In a steady-state equilibrium, normalizing $A = 1$,

(i) An increase in $r$ will cause no change in $q$, a decrease in $\phi$, and an increase in $\kappa$.

(ii) An increase in $\sigma$ will cause no change in $q$, an increase in $\phi$, and a decrease in $\kappa$.

Proposition 4.8 gives the comparative statics for the rate of time preference and matching probability. As agents become more patient, the seigniorage-maximizing sovereign will choose a lower amount of commodity to put in the money. Seigniorage-maximizing sovereigns will also put less commodity in the money as matching probabilities increase. This can be interpreted as markets growing thicker and more integrated. As the probability a peasant can find a match grows, the usefulness of money grows. The liquidity value of money grows accordingly, and the amount of commodity required in the money for peasants to hold it falls.

Both of these results can be viewed as movement starting within the Fiat infeasible region of figure 3. An increase in $\sigma$ is a movement to the right, toward the Fiat feasible region. As fiat becomes more feasible, the seigniorage-maximizing sovereign makes the currency more fiat and less commodity. The bound, $r/(1 + r)$ decreases as $r$ decreases, so for any point in Fiat infeasible the boundary of feasibility for fiat money comes closer, and the seigniorage-maximizing sovereign reduces the commodity-ness of the currency in response.

### 4.4 Money Growth and Optimal Monetary Policy

The environment, as discussed above, does not consider money growth. This section establishes a connection between standard money growth in the Lagos-Wright framework and the choice of return on the asset in the seigniorage model above. It will be shown that the optimal monetary policy corresponds to a choice of zero seigniorage, but not zero dividend.
Consider the standard Lagos-Wright fiat money framework with a constant gross rate of money growth, $\gamma$. The buyers face a first-order condition that determines their asset holdings as follows:

$$\frac{-\gamma}{\beta} - \frac{\beta}{\beta} + \sigma \theta L(q) \leq 0.$$

This is analogous to (12), and these conditions are equivalent for

$$\gamma = \frac{\phi}{\phi + \kappa}.$$  

(27)

The range of $\gamma$ is $[\beta, 1]$ - if $\gamma < \beta$, then the demand for money would be unbounded and if $\gamma > \beta$, then money would not be valued sufficiently to purchase any DM good. In the environment described above, every $(\phi, \kappa)$ pair can be replicated in a pure fiat environment with a gross growth rate of money, $\gamma = \phi/(\phi + \kappa)$. However, the point of the environment above is that sometimes the primitives, $(r, \sigma, \theta)$, of an environment make fiat infeasible. If fiat were infeasible, the question then is, given a target gross growth rate of money, $\gamma$, and a stationary price of money, $\phi$, what amount of dividend, $\kappa$, generates an analogous economy?

First, note that the range of $\gamma$ can be divided:

$$\gamma \in \begin{cases} 
[\beta, 1) & : \text{deflation} \\
\{1\} & : \text{no money growth} \\
(1, \beta [1 + \sigma \theta/(1 - \theta)]) & : \text{inflation}.
\end{cases}$$

(28)

From (27) it is immediately obvious

$$> \quad : \quad \text{deflation}$$

$$\kappa = 0 \quad : \quad \text{no money growth}$$

$$< \quad : \quad \text{inflation}.$$  

(29)

If the dividend could be negative - if the sovereign could collect a tax on the currency - this would be equivalent to the inflation tax in a fiat money model with gross money growth greater than one. A positive dividend corresponds to deflation. The Friedman rule in the
Lagos-Wright framework is well-known to be $\gamma = \beta$. From this result, the optimal amount of dividend that corresponds to the optimal monetary policy is $\kappa = r\phi$ for a given $\phi$.

Returning to the original question, given $(r, \sigma, \theta)$, assume fiat is infeasible. From Proposition 4.6, this implies $r > \sigma\theta/(1 - \theta)$. Furthermore, this implies

$$\beta \left(1 + \frac{\sigma\theta}{1 - \theta}\right) < 1.$$  \hfill (30)

The left-hand side of this inequality is the upperbound of $\gamma$ such that money is still valued. When this upperbound is less than one, fiat money is only valued under deflation. In the commodity money environment, this requires $\kappa > 0$ - specifically, $\kappa = \phi(1 - \gamma)/\gamma$.

5 Extensions

5.1 Convertibility

The following section investigates the theoretical underpinnings of a type of commodity money observed in history: convertible money. Convertible money is often viewed as the stepping stone to fiat. Paper money is issued backed by some amount of commodity money. If the money is convertible, the paper money can be redeemed in commodity money upon demand. During the free banking era in the United States (1837-1864), individual banks were granted the right to print their own paper money, so long as that money were properly backed by state bonds and redeemable for their gold value. During crises it sometimes became necessary for the banks to suspend their convertibility feature.

From the perspective of the history of monetary and economic thought, this discussion resolves a long debate between what are frequently called the quantity theory and real bills doctrine. Real bills has come in many forms but the latest incarnation is a backing theory of money, the theory that money is valued because there is an asset backing it. In the following section I discuss models of convertible money to understand the role convertibility itself plays. I find that a currency that circulates (formally: is not redeemed immediately) does not resemble commodity money as modeled in the previous section. If fiat were not valued, then convertible money can circulate if an agent is required to search for an agent who will redeem.
Figure 6: The graph of the demand for one-period Lucas Trees. Note that the fundamental value is $\kappa/(1 + r)$ since the asset cannot be converted the period it is issued. The maximum price is $\kappa F$, where $F = \left[1 + \sigma\theta/(1 - \theta)\right]/(1 + r)$. When $F \geq 1$, fiat is feasible; when $F < 1$ fiat is not feasible. If this asset were immediately convertible for $\kappa$, then if fiat were not feasible as money, neither would this asset serve as money, for the upperbound of the price would fall below $\kappa$.

The following subsection is structured as follows. First, I create a benchmark with one-period Lucas trees. These assets cannot be redeemed upon issue and are completely redeemed the following period. Second, I provide a model of convertible money where it is costless to redeem. Finally, I model convertible money that is costly to redeem and consider comparative statics.

5.1.1 One-Period Lucas Tree

The environment is the same as in the benchmark except for the particulars of money. Money is issued as one-period Lucas trees that pay a dividend the period following their issue. In this way, the entirety of the money supply is redeemed every period for $\kappa$, which means the fundamental value of the asset at sale is $\kappa/(1 + r)$. As in the previous model, sellers do not benefit from trade by bringing money into a match. If the price of the asset is above the fundamental, there will be no seller-driven demand for it. Below are the Bellman equations
characterizing the value functions of the buyer in CM and DM, respectively.

\[
W^b(a) = \max_{x,a' \geq 0} [x + \beta V^b(a')]
\]

\[
s.t. \ x + a' \phi \leq a\kappa
\]

\[
V^b(a) = \sigma[u(q) + W^b(a - d)] + (1 - \sigma)W^b(a)
\]

The bargaining problem looks similar to the benchmark, except the value of the asset in the following period is \(\kappa\), not \(\kappa + \phi\).

\[
\max \theta [u(q) - c(q)]
\]

\[
s.t. (1 - \theta)u(q) + \theta c(q) \leq \kappa a
\]

Applying backwards induction, the following condition arises between the fundamental value and price of money:

\[
\phi \geq \kappa \frac{1 + \sigma \theta \mathcal{L}(q)}{1 + r} \geq \frac{\kappa}{1 + r}.
\]

Note that as \(A \to 0\), \(q \to 0\), \(\mathcal{L}(q) \to 1/(1 - \theta)\), so

\[
\phi \to \kappa \frac{1 + \frac{\sigma \theta}{1 - \theta}}{1 + r}
\]

Relating this kind of commodity money to the benchmark, the questions are, "If fiat money is not valued, will commodity money be valued? If so, does the level of commodity in the money matter?" From the benchmark we know fiat is not feasible if

\[
\frac{\sigma \theta}{1 - \theta} < r.
\]

If fiat is not feasible, then from above, the bounds on \(\phi\) are

\[
\kappa \frac{1 + \frac{\sigma \theta}{1 - \theta}}{1 + r} \geq \phi \geq \frac{\kappa}{1 + r}
\]

and the upperbound is less than \(\kappa\).
Setting a price floor at $\kappa$, as allowing convertibility for $\kappa$ would do, would mean the currency would only be valued if fiat money were also valued. Note that the level of $\kappa$ does not matter so long as $\kappa > 0$.

For these one-period Lucas trees, the price can fall below $\kappa$ because they are only redeemable in the following period. In a situation with convertible money, redemption is available at any time. This means price cannot fall below $\kappa$, or an unbounded quantity would be demanded at $\phi < \kappa$ and immediately redeemed for unbounded profit. In order to avoid a currency where convertibility does not provide any benefit over fiat money, one could restrict money not to be convertible until the following period, as in the model above.

### 5.1.2 Costless Redemption Convertibility

Assume that the asset provides no dividend, but can be converted to $\kappa$ units of general good during any CM. Let $W^b(a)$ denote the lifetime expected utility of a buyer in the CM holding $a$ perpetuities in units of the numéraire good. Let $V^b$ be its value function in the DM.

\[
W^b(a) = \max_{x,a' \geq 0, \omega \in [0,1]} \left[ x + \beta V^b(a') \right] \\
\text{s.t. } x + a' \phi = a[(1 - \omega)\phi + \omega \kappa].
\]

The difference between this program and the problem of the buyer in the CM in the original version is the right-hand side of the budget constraint. The buyer decides what fraction of assets, $\omega$, he wants to convert into $\kappa$ units of the general good. The rest are sold at price $\phi$. The constraint can be substituted into the value function:

\[
W^b(a) = \max_{a' \geq 0, \omega \in [0,1]} (-a' \phi + a[(1 - \omega)\phi + \omega \kappa] + \beta V^b(a')).
\]

Note that the choice of $\omega$ is independent of the choice of $a'$:

\[
W^b(a) = a \max_{\omega \in [0,1]} [(1 - \omega)\phi + \omega \kappa] + \max_{a' \geq 0} [-a' \phi + \beta V^b(a')].
\]

This is a linear program, so the buyer will choose $\omega = 0$ if $\phi > \kappa$, $\omega = 1$ if $\phi < \kappa$ and may choose any $\omega \in [0,1]$ if $\phi = \kappa$. All buyers are symmetric. This implies the case where $\phi < \kappa$ will immediately lead to all currency being converted immediately - the asset will disappear.
A stationary solution with $\phi > 0$ is only possible for $\phi \geq \kappa$.

The interpretation of convertibility can either be that the sovereign stands ready to convert token currency for its backed value, $\kappa$, or that the currency has an intrinsic value that can be obtained through a costless melting process.\footnote{Historically, the melting process was subject to a cost called seigniorage.}

\[ V^b(a) = \sigma \{ u(q) - d \max_{\omega \in [0,1]} [(1 - \omega)\phi + \omega \kappa] \} + W^b(a) \] \tag{35}

The choice of assets is determined by the following problem for the buyer:

\[ \max_{a \geq 0} \{ -a \phi r + a \max_{\omega \in [0,1]} \omega (\kappa - \phi) + \sigma \theta [u(q) - c(q)] \} \] \tag{36}

Stationarity in an environment without money growth requires either no money to be held or none of it to be converted. If money cannot be converted, the option value will not be exercised in equilibrium. Any equilibrium where the option value is not exercised but money is valued must be one where fiat is feasible.

Convertible currency does not give you the same properties as a bond that pays dividends each period.

### 5.1.3 Costly Redemption Convertibility

It is less than realistic to assume that paper money could be converted costlessly. For instance, during the Napoleonic wars, Great Britain suspended convertibility of paper money. At this time, the Bank of England monopolized paper money issue within the city of London. To convert your paper money you needed to go directly to the Bank of England. Another example comes from the Free Banking era in the United States (1837-1864). During this era, state-chartered banks were given authority to issue currency under the requirement they redeem their notes for gold at face value. Gorton (1999) and Ales et al (2008) provide history and theory for the discounting of bank notes from this era. Notes circulated below their face value; it was costly to locate the note issuers to redeem for the face value. Famously, Nicholas Biddle, president of the Second Bank of the United States, used the threat of sending large quantities of notes for redemption at state banks to keep other banks in line with his policies.
A model for convertibility ought to take these observations into account. Whereas the previous model of convertible money does not provide any benefit over fiat, the following model does. As in the one-period Lucas tree model, an agent cannot redeem their paper money for the backed value immediately. Agents search and randomly match with currency issuers as well as sellers in the decentralized market. Currency issuers must redeem for face value \( \kappa \) if the agent so desires.

This way of modeling convertibility, where agents need to endure search and matching costs, gives rise to a mode where the value of money can, in equilibrium, fall below the face value of the currency. Suspension of convertibility is modeled as a decrease in the match rate with currency issuers. This is in contrast to the previous model where suspension is modeled as eliminating the option to redeem in the centralized market completely.

\[
W^b(a) = \max_{x,a'} [x + \beta V^b(a')] 
\]

\[
s.t. \ x + \phi a' \leq a \phi 
\]

\[
\implies W^b(a) = a \phi + \max_{a'} [-\phi a' + \beta V^b(a')] 
\]

\[
V^b(a) = \sigma[u(q) + W^b(a - d)] + \omega[d_g \kappa + W^b(a - d_g)] + (1 - \sigma - \omega)W^b(a) 
\]

\[
\implies V^b(a) = \sigma[u(q) - \phi d] + \omega[d_g \kappa - \phi d_g] + W^b(a) 
\]

When an agent carrying the money good encounters the currency issuer in the decentralized market, they may redeem \( d_g \leq a \) of their assets for face value.

In a match, buyers and sellers bargain following the proportional bargaining rule. The buyer’s bargaining problem is:

\[
\max_{q \geq 0} [u(q) - d \phi] 
\]

\[
s.t. \ u(q) - d \phi = \frac{\theta}{1 - \theta} [d \phi - c(q)] 
\]

\[
d \leq a. 
\]

Proportional bargaining between buyer and seller implies \( z(q) = d \phi \leq a \phi \). Applying
backwards induction to solve for the buyer’s problem yields

\[
\max_{a \geq 0, \pi \in [0, 1]} \left\{ -r\phi a + \sigma \theta[u(q) - c(q)] + \omega d_g(\kappa - \phi) \right\} \tag{45}
\]

\[
s.t. \ d_g = a \pi. \tag{46}
\]

Note that \( \pi \) represents the choice of agents whether or not to redeem, where \( \pi = 0 \) if agents do not wish to redeem, \( \pi = 1 \) if agents wish to redeem and \( \pi \in [0, 1] \) if agents are indifferent. The interesting case to examine, as before, is when \( \kappa > \phi \) and fiat is no longer feasible, \( \sigma \theta \mathcal{L}(q) - r < 0 \). In this case,

\[
\phi = \frac{\omega \kappa}{\omega - [\sigma \theta \mathcal{L}(q) - r]}
\]

As in the benchmark, the amount of commodity-ness in the currency is determined by a seigniorage maximizing sovereign. Unlike the benchmark, the sovereign only needs to redeem the currency under the restrictions that agents must meet them in the DM and agents must want to redeem their currency. Then the sovereign’s problem is

\[
\max A[\phi + \omega \pi \sum_{t=1}^{\infty} \beta^t(\phi - \kappa)] \tag{47}
\]

\[
s.t. \ A \in A^d(\phi), \tag{48}
\]

where \( A^d(\phi) \) is determined by \( \text{(45)} \).

In the case \( \kappa > \phi, \pi = 1 \) and the sovereign’s problem reduces to

\[
\max \ \theta z(q) \mathcal{L}(q), \tag{49}
\]

which is the same as in the benchmark. This means the sovereign will choose an amount of commodity in the currency to target a quantity of trade in the DM regardless of \( \sigma \) and \( r \) - parameters denoting thickness of markets and patience of the agents.
Definition 2. A steady-state equilibrium is a sequence of \( \{q_t, \phi_t, d_t, d_{gt}, \pi_t, a_t, A_t, \kappa_t, \omega_t\}_{t=0}^{\infty} \) such that

1. \( q_t = q, \phi_t = \phi, d_t = d, d_{gt} = d_g, \pi_t = \pi, a_t = a, A_t = A, \kappa_t = \kappa, \omega_t = \omega \) \( \forall t \geq 0 \),

2. agents bargain to determine \( \{q,d\} \) (42), (43), (44),

3. agents choose \( \{a,d_g,\pi\} \) to maximize utility (45), (46)

4. the sovereign chooses \( \{A,\kappa,\omega\} \) to maximize seigniorage subject to the demand for assets (47), (48),

5. prices follow \( z(q) \equiv \theta c(q) + (1 - \theta) u(q) = a \phi \),

6. and markets clear \( A = a \).

Proposition 5.1. In a steady-state equilibrium, normalizing \( A \kappa = 1 \),

(i) An increase in \( r \) will cause no change in \( q \), a decrease in \( \phi \), and an increase in \( \omega \).

(ii) An increase in \( \sigma \) will cause no change in \( q \), an increase in \( \phi \), and a decrease in \( \omega \).

While this fits the long-run pattern of governments making their currency more and more fiat, making convertible money less easily convertible over time, it does not explain temporary suspensions of convertibility. A temporary suspension might be considered an exogenous increase in \( r \) - agents become suddenly less patient. In the model, the government would respond to the impatience by making it easier to convert money. In reality, a suspension meant the opposite.

5.2 Oligarchy

Assume that instead of a single sovereign, there is instead an oligarchy, \( \alpha \in [0,1] \) fraction of buyers that, as we assumed with the sovereign, can commit to paying a dividend, \( \kappa \), on a perpetuity sold at the beginning of time. The peasants’ problems do not change, but the oligarchy’s problem becomes

\[
\max_{A \geq 0, \kappa \geq 0, \phi \geq 0} [A(\phi - \phi^*) + \alpha W^b(0)]
\]

\[\text{s.t. } A \in A^d(\phi).\] (50) (51)
In the problem with a single sovereign that does not consume in the DM, the scarcity of the asset increases its liquidity value, and hence seigniorage. The oligarchy must face the tradeoff between seigniorage and trade surplus. The sovereign’s problem can be rewritten as

$$\max_{\kappa \geq 0, \phi \geq 0} \{(1 - \alpha)z(q)\mathcal{L}(q) + \alpha[u(q) - c(q)]\}.$$  \hfill (52)

**Definition 3.** A steady-state equilibrium is a sequence of \(\{q_t, \phi_t, d_t, a_t, A_t, \kappa_t\}_{t=0}^{\infty}\) such that

1. \(q_t = q, \phi_t = \phi, d_t = d, a_t = a, A_t = A, \kappa_t = \kappa \ \forall t \geq 0\),

2. agents bargain to determine \(\{q, d\}\) \cite{oligarchy}, \cite{bargain}, \cite{determine},

3. agents choose \(a\) to maximize utility \cite{utility},

4. the sovereign chooses \(\{A, \kappa\}\) to maximize seigniorage subject to the demand for assets \cite{demand}, \cite{demand2},

5. prices follow \(z(q) \equiv \theta c(q) + (1 - \theta)u(q) = a(\phi + \kappa),\)

6. and markets clear \(A = a\).

Following the initial formulation, \(A\) is normalized to one. This program is concave if \(z(q)\mathcal{L}(q)\) is concave. The oligarchy’s problem is once again rewritten as if they were choosing \(q\). The DM trade surplus term will lead to the choice of a greater \(q\).

**Proposition 5.2.** As \(\alpha\) increases, equilibrium \(q\) increases and \(\kappa\) increases.

**Corollary 5.3.** As \(\alpha \to 1\), \(q \to q^\ast\).

As those with a greater interest in the value of currency for its use in trade gain influence over the sovereignty, the oligarchy approaches a welfare-maximizing planner. As established above, greater dividend in the currency is analogous to more deflation in a fiat money economy. By analogy, as the oligarchy grows, the Friedman-rule equivalent level of dividend is approached, where seigniorage is zero and there is sufficient currency that trade surplus is maximized.
5.3 Two Currencies

Historically, gold and silver were both used as commodity money. The model as previously presented can be interpreted as an abstraction from the kind of metal actually used in the money. What follows is a step closer to reality, allowing for two commodity monies. By explicitly modeling two currencies, the model gives predictions regarding the exchange rate.

Consider the same environment for two commodity monies, red and blue. A buyer’s value function in the CM is:

\[
W(a_b, a_r) = a_b(\phi_b + \kappa_b) + a_r(\phi_r + \kappa_r) + \max_{a'_b, a'_r}\{-\phi_b a'_b - \phi_r a'_r + \beta V(a'_b, a'_r)}
\]

where the \(r\) subscript denotes red and \(b\) denotes blue. The buyer’s value function in the DM is:

\[
V(a_b, a_r) = \sigma\{u[q(a_b, a_r)] - d_b(q)(\phi_b + \kappa_b) - d_r(q)(\phi_r + \kappa_r)\} + W(a_b, a_r).
\]

The buyer’s bargaining problem is

\[
\max_{q \geq 0} [u(q) - d_r(\phi_r + \kappa_r) - d_b(\phi_b + \kappa_b)]
\]

s.t. \(u(q) - d_r(\phi_r + \kappa_r) - d_b(\phi_b + \kappa_b) = \frac{\theta}{1 - \theta}[d_r(\phi_r + \kappa_r) + d_b(\phi_b + \kappa_b) - c(q)]\)

\(d_r \leq a_r\)  
\(d_b \leq a_b.\)

The buyer’s choice of assets can be written as:

\[
\max_{a_b, a_r} - a_b r\left(\phi_b - \frac{\kappa_b}{r}\right) - a_r r\left(\phi_r - \frac{\kappa_r}{r}\right) + \sigma \theta[u(q) - c(q)]
\]

s.t. \(\theta c(q) + (1 - \theta) u(q) = a_b(\phi_b + \kappa_b) + a_r(\phi_r + \kappa_r).\)

Assuming an interior solution:

\[
\frac{\sigma \theta[u'(q) - c'(q)]}{\theta c'(q) + (1 - \theta) u'(q)} = \frac{r \phi_b - \frac{\kappa_b}{r}}{\phi_b + \kappa_b} = \frac{r \phi_r - \frac{\kappa_r}{r}}{\phi_r + \kappa_r}
\]
Assuming the following relation between the prices and dividends of the two currencies,

\[ \phi_b = \delta \phi_r \]  
\[ \kappa_b = \epsilon \kappa_r, \]

and using the previous equality gives

\[ \frac{\delta \phi_r - \epsilon \kappa_r}{\delta \phi_r + \epsilon \kappa_r} = \frac{\phi_r - \kappa_r}{\phi_r + \kappa_r}. \]  

(64)

It is obvious this equation will be true when \( \delta = \epsilon \).

From this the exchange rate is derived:

\[ \phi_b = \frac{\kappa_b}{\kappa_r} \phi_r. \]  

(65)

If this exchange rate does not hold, the overpriced money will stop circulating. An incident from history that illustrates this prediction comes from Britain. As master of the mint, Sir Isaac Newton in 1717 set a high silver-price of gold. This price led to silver flowing out of the country, effectively creating the first gold standard in Britain.

The sovereign faces the following program:

\[
\max_{A_r \geq 0, \kappa_r \geq 0, \phi_r \geq 0, A_b \geq 0, \kappa_b \geq 0, \phi_b \geq 0} A_r (\phi_r - \phi_r^*) + A_b (\phi_b - \phi_b^*) \\
\text{s.t. } A_r \in A_r^d(\phi_r) \\
A_b \in A_b^d(\phi_b).
\]

(66)

(67)

(68)

This also can be rewritten as

\[
\max \theta z(q) L(q). 
\]

(69)

**Definition 4.** A steady-state equilibrium is a sequence of \( \{q_t, \phi_{rt}, d_{rt}, a_{rt}, A_{rt}, \kappa_{rt}, \phi_{bt}, d_{bt}, a_{bt}, A_{bt}, \kappa_{bt}\}_{t=0}^{\infty} \) such that

1. \( q_t = q, \phi_{rt} = \phi_r, d_{rt} = d_r, a_{rt} = a_r, A_{rt} = A_r, \kappa_{rt} = \kappa_r, \phi_{bt} = \phi_b, d_{bt} = d_b, a_{bt} = a_b, A_{bt} = A_b, \kappa_{bt} = \kappa_b \) \( \forall t \geq 0, \)
2. agents bargain to determine \{q, d_r, d_b\} \[55\], \[56\], \[57\], \[58\],

3. agents choose \{a_r, a_b\} to maximize utility \[59\], \[60\]

4. the sovereign chooses \{A_r, k_r, A_b, k_b\} to maximize seigniorage subject to the demand for assets \[66\], \[67\], \[68\],

5. prices follow \[z(q) \equiv \theta c(q) + (1 - \theta)u(q) = a_r(\phi_r + k_r) + a_b(\phi_b + k_b)\],

6. and markets clear \[A_r = a_r, A_b = a_b\].

In the situation where there is no limit to the availability of the commodities used to create the currencies, the sovereign is indifferent between red and blue. The total amount of commodity-ness in the currency is used to target the \[q\] that maximizes seigniorage and the sovereign has no preference regarding what mix of currency is used to do that.

### 5.4 Money and Credit

Typical assumptions, such as anonymity and limited record keeping, are required to make money essential in a monetary model. Historically, these assumptions are strange on a village-level. Peasants that live together know each other. They have long-lasting relationships. Reputations matter in the village. These are conditions that can allow credit - formal or informal - to arise. For money to serve a role, some interactions need to be anonymous. The historical analogue can be considered the faire. In a faire merchants come to a village to sell their wares before moving on to the next town. These trading opportunities cannot make use of credit, as there is no record keeping and the merchants may not pass through that town again. To capture this aspect of trade, I model the village and the faire.

There are two decentralized markets, the village and the faire. At the end of the CM agents go to the village with probability \[\epsilon\] and go to the faire with probability \[1 - \epsilon\]. At the faire agents search and match with agents whom they will meet again with probability zero. This should be considered the same as the DM in the previous versions. The village is different in that credit through public record keeping is available. Individuals may issue one-period IOUs (that fully depreciate at the end of the CM) to agents they meet in the village DM. Because these notes fully depreciate at the end of the following CM they will not circulate across periods - they will either be redeemed in the CM or their issuers will
default and the notes will fully depreciate. If an agent defaults they will be punished by never being allowed to use credit again.

![Diagram of CM, Village, and Faire](image)

Figure 7: With probability $\epsilon$ agents go to the village, where credit is possible, and may match with probability $\sigma_1$. With probability $1 - \epsilon$ they go to the faire, where they match with probability $\sigma_2$.

The buyer’s value functions in the CM, when they have access to credit (assuming no default) and when they are being punished, respectively, are as follows:

\[
W(a, b) = a(\phi + \kappa) - b + \max_{a' \geq 0} \{-a'\phi + \beta[\epsilon V(a') + (1 - \epsilon)F(a')]\}
\]

\[
W_p(a) = a(\phi + \kappa) + \max_{a' \geq 0} \{-a'\phi + \beta[\epsilon V_p(a') + (1 - \epsilon)F_p(a')]\},
\]

where the subscript $p$ denotes a buyer who will be punished by not having access to credit, $V$ represents the value function for a buyer in the village DM and $F$ represents the value function for a buyer in the faire DM. Note $W(a, b)$ is linear in $a$ and in $b$ and $W_p(a)$ is linear in $a$. The DM value functions are:

\[
V(a) = \sigma_1[u(x) + W(a, b)] + (1 - \sigma_1)W(a, 0)
\]

\[
V_p(a) = \sigma_1[u(q) + W_p(a - d)] + (1 - \sigma_1)W_p(a)
\]

\[
F(a) = \sigma_2[u(q) + W(a - d, 0)] + (1 - \sigma_2)W(a, 0)
\]

\[
F_p(a) = \sigma_2[u(q) + W_p(a - d)] + (1 - \sigma_2)W_p(a).
\]

Again, the subscript, $p$ denotes a punished buyer, $\sigma_1$ denotes the match probability in the village and $\sigma_2$ denotes the match probability in the faire.

When a punished buyer matches with a seller in the village, the buyer can only trade
with money. Assuming a proportional bargaining rule, the buyer’s problem is exactly the same as a matched buyer faces at the faire, whether punished or unpunished. When an unpunished buyer matches with a seller in the village, the buyer can use credit:

\[
\begin{align*}
\max_{x,b,d} & \quad u(x) - b - d(\phi + \kappa) \\
\text{s.t.} \quad & u(x) - b - d(\phi + \kappa) = \frac{\theta}{1-\theta}[b + d(\phi + \kappa) - c(x)] \\
& d \leq a \\
& b \leq \bar{b},
\end{align*}
\]

where \(\bar{b}\) is the endogenous borrowing limit. Assuming a proportional bargaining rule, the punished buyer’s value function in the CM can be rewritten as

\[
W_p(a) = a(\phi + \kappa) + \max_{a' \geq 0} \{-a'\phi + \beta \theta \epsilon \sigma_1[u(q) - c(q)] + a'(\phi + \kappa) + W_p(0)\}
\]

\[
rW_p(0) = \max_a \{-r(\phi - \phi^*)a + \theta \epsilon \sigma_1[u(q) - c(q)]\}.
\]

Above, \(\sigma_\epsilon\) denotes \(\epsilon \sigma_1 + (1-\epsilon)\sigma_2\), the expected match probability before knowing whether the buyer is going into faire or village. Note that \(W(a,b)\) can be rewritten

\[
W(a,b) = a(\phi + \kappa) - b + \max_{a' \geq 0} [-a'\phi + \beta \theta \epsilon \sigma_1[u(x) - c(x)] + \\
\theta(1-\epsilon)\sigma_2[u(q) - c(q)] + a'(\phi + \kappa) + W(0,0)].
\]

This provides an endogenous borrowing limit based on the no-default constraint:

\[
\begin{align*}
b & \leq \bar{b} = \beta \max_{a' \geq 0} \{-r(\phi - \phi^*)a' + \theta(1-\epsilon)\sigma_2[u(q) - c(q)] + \\
& \theta \epsilon \sigma_1[u(x) - c(x)]\} + \beta W(0,0) - W_p(0)
\end{align*}
\]

\[
b + d(\phi + \kappa) = \theta c(x) + (1-\theta)u(x).
\]

**Definition 5.** A steady-state equilibrium is a sequence of \(\{q_t, x_t, \phi_t, d_t, a_t, b_t, \bar{b}_t, A_t, \kappa_t\}_{t=0}^\infty\) such that

1. \(q_t = q, x_t = x, \phi_t = \phi, d_t = d, a_t = a, b_t = b, \bar{b}_t = \bar{b}, A_t = A, \kappa_t = \kappa \forall t \geq 0,\)
2. agents bargain to determine \( \{q, d, b\} \) \((6), (7), (8), (76), (77), (78), (79)\),

3. agents choose \( a \) to maximize utility \((82)\).

4. the sovereign chooses \( \{A, \kappa\} \) to maximize seigniorage subject to the demand for assets \((21), (22)\),

5. prices follow \( z(q) \equiv \theta c(q) + (1 - \theta) u(q) = a(\phi + \kappa) + \bar{b} \),

6. the no-default constraint holds \((83)\),

7. and markets clear \( A = a \).

The amount the buyer can purchase when in the village is at least as much as he can purchase in the faire, as he will bring the same amount \( a \) into either meeting, not knowing which decentralized market he will attend next. This is true because the credit available in the village cannot decrease the amount produced in a match. The endogenous borrowing limit is increasing in \( \sigma_1 \) - the greater your match probability within the village, the more credit is available. As \( x \geq q, \) and \( x \leq q^*; \) \( u(x) - c(x) \geq u(q) - c(q) \), so the borrowing limit is increasing in \( \epsilon \), the probability of going to the village.

The interpretation here is in line with the history. As the number of trades which can be supported by reputation and punishment, as modeled by the village credit, decreases, either because the probability of finding what you need within your own village decreases, or as the probability of doing business in the village at all decreases relative to trading with merchants - strangers - as modeled by the faire, the borrowing limit falls and credit is used less. This represents a transition from credit to currency as markets grow thicker and more integrated.

6 Examples from History

6.1 Aztec Money

Weatherford's *The History of Money* provides some stylized facts about the early money of Aztec culture. One commodity used for money was the cacao seed, also called beans. Pods from the cacao tree bore seeds which, after preservation, could last for months. These seeds
were eventually ground up and consumed as part of a chocolate drink. In the mean time, preserved cacao seeds were used as a medium of exchange. Weatherford reports cacao was used to purchase "fruits and vegetables such as corn, tomatoes, chilies, squash, chayotes, and peanuts; jewelry made of gold, silver, jade, and turquoise; manufactured goods such as sandals, clothing, feathered capes, cotton padded armor, weapons, pottery, and baskets; meats such as fish, venison, duck; and specialty goods such as alcohol and slaves." Despite this variety of goods which accepted cacao in exchange, there were a couple quirks to their usage. Firstly, the cacao was used as small change in barter agreements with indivisible goods. Secondly, the marketplace was a specially designated area monitored by government officials. In these marketplaces, prices were regulated and enforced by the threat of death for serious offenses. A final note of interest regarding cacao as money: it was counterfeited. The seeds could be shelled, essentially, and then filled with mud. These counterfeit seeds would be mixed with genuine cacao in sales.

In what follows I provide a model to try to match some stylized facts from the history of Aztec cacao money:

1. there was an essentially "free-minting" policy regarding the creation of new cacao money; if you were willing to pay the physical cost to preserve more cacao, you could literally make money;

2. preserved cacao depreciated after a few months, at which point it was consumed;

3. prices in the centralized market were regulated by the government.

Aspects that are interesting but I will not attempt to capture with this model include:

- cacao used as small change in a world of indivisible goods dominated by barter exchange;
- counterfeiting.

The framework is essentially the same as the benchmark provided above, with some important differences. Agents may choose to preserve new cacao at a convex cost \( \pi(\cdot) \) in the CM. Agents may choose to consume cacao for linear utility \( \kappa \). Cacao depreciates and

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\(^9\text{pp. 18-19}\)

\(^{10}\text{pp. 17-18}\)
is only good for consumption, not as a medium of exchange, at rate $\delta$. Let $\mu$ denote the newly preserved cacao in a period, let $c$ denote the amount of cacao chosen by an agent to be consumed in a period over and above the $\delta$ proportion of holdings that must be consumed due to depreciation. As usual, $\gamma$ represents the gross growth rate of money, but this is not directly controlled by the government, as there are no barriers to the production of new cacao for agents. Instead, the government may control $\phi$, the CM-good price of cacao.

Buyers in the CM enter with a real balance of cacao, $z$, and choose how much CM good to produce or consume, $x$, how much of their cacao holdings to consume, $c$, how much new cacao to produce, $\mu$, and how much to accumulate to take to the next DM, $z'$.  

$$W^b(m) = \max_{x,c,\mu,m'} \left[ x + \kappa(\delta m + c) - \pi(\mu) + \beta V^b(m') \right]$$  \hspace{1cm} (85)  

s.t. $x + \phi m' = (1 - \delta)m \phi + [\mu - c] \phi$.  \hspace{1cm} (86)  

This can be consolidated:

$$W^b(m) = [(1 - \delta)\phi + \kappa \delta]m + \max_{c} \left[ -\phi c + \kappa c \right] + \max_{\mu} \left[ \phi\mu - \pi(\mu) \right] + \max_{m'} \left[ -\phi m' + \beta V^b(m') \right].$$

The amount of newly created cacao each period, $\mu$, is determined by the CM-good price of cacao, $\phi$, and the cost of preserving cacao, $\pi(\cdot)$. The amount consumed, $c$ is determined by the CM-good price of cacao, the utility of consumption, and how much the agent is already consuming due to depreciation. The gross growth rate of money, $\gamma$, which is taken as given by each agent, is the ratio of the sum of the undepreciated cacao, $\delta m$, and the newly preserved cacao, $\mu$, less the consumed cacao, $c$, to the previous amount of cacao, $m$. It is worth noting that if $\phi < \kappa$ agents will demand unbounded amounts of cacao for pure consumption. Cacao as a currency would disappear. If $\phi > \kappa$ then $c^* = 0$ and the only cacao consumed will be amount that can no longer be used as money due to depreciation. From here on out it will be assumed $\phi > \kappa$. As in the benchmark model, $W^b(m)$ is linear in $m$ and the choice of $m'$ is independent of $m$.

The buyer’s value function in the DM is

$$V^b(m) = \sigma \{ u(q) - [(1 - \delta)\phi + \delta \kappa]d \} + W^b(m).$$
Proportional bargaining within a match yields

\[ d = \frac{z(q)}{(1 - \delta)\phi + \delta \kappa}. \]  

(87)

Backwards induction and the first order condition yields a condition that determines how buyers choose cacao money holdings:

\[ (i + \delta)[\phi - \kappa \frac{\delta}{i + \delta}] = \sigma \theta L(q)[(1 - \delta)\phi + \delta \kappa]. \]

The gross growth rate of money is

\[ \gamma = \frac{(1 - \delta)m + 2\mu}{m}, \]

where the 2 arises from the fact both buyers and sellers will preserve more cacao to use as money in the CM. The decision to create new money is determined by the first-order condition, \( \phi = \pi'(\mu) \).

7 Conclusion

The reason is took so long for fiat money to be the norm is that people decide whether they wish to hold a good as money depending on market thickness, competitiveness of markets, and their patience. There has been a long-run trend towards market integration, lower trade surpluses to sellers, and increased life expectancy, which have affected the feasibility of fiat. Self-interested sovereigns take the behavior of their subjects into account when attempting to maximize seigniorage. To this end, seigniorage is maximized when fiat is infeasible by adding intrinsic value - historically, in the form of precious metal or promises of convertibility - to the currency. Sovereigns are willing to pay this price because they can manufacture a scarcity in the money that gives it a liquidity premium. The premium is then extracted as seigniorage.
Figure 8: The Aztec government sets $\phi$ and thereby controls the gross growth rate of cacao money. As $\mu > \delta m + c$ in the figure, this represents an economy with inflation.
References


Del Mar, Alexander, “A history of money in ancient countries, from the earliest times to the present,” 1968.


Appendix

Proof of Lemma 4.2 First, it is important to note that

\[
0 \leq c(q) \leq \theta c(q) + (1 - \theta)u(q) \leq u(q) \quad \forall q \in [0, q^*] \\
0 \leq c'(q) \leq \theta c'(q) + (1 - \theta)u'(q) \leq u'(q) \quad \forall q \in [0, q^*] \\
u''(q) \leq \theta c''(q) + (1 - \theta)u''(q) \leq c''(q) \quad \forall q \in [0, q^*].
\]

For \( q \in [0, q^*] \), by Assumption 4.1

\[
u''(q) - c''(q) < \frac{2c''(q)[u''(q)c'(q) - c''(q)u'(q)]}{[c'(q)]^3}
\]
\[\Rightarrow u''(q)c'(q) - c''(q)c'(q) < \frac{2c''(q)[u''(q)c'(q) - c''(q)u'(q)]}{[c'(q)]^2}
\]
\[\Rightarrow u''(q)c'(q) - c''(q)u'(q) < \frac{2c''(q)[u''(q)c'(q) - c''(q)u'(q)]}{[c'(q)]^2}
\]
\[\Rightarrow \frac{[u''(q)c'(q) - c''(q)u'(q)][c'(q)]^2}{[\theta c'(q) + (1 - \theta)u'(q)]^4} - \frac{2c''(q)[u''(q)c'(q) - c''(q)u'(q)]}{[\theta c'(q) + (1 - \theta)u'(q)]^4} < 0
\]
\[\Rightarrow \mathcal{L}''(q) < 0.
\]

The third line follows from \( c''(q) > 0 \) for all \( q \in [0, q^*] \). This line implies \( u''(q)c'(q) - c''(q)u'(q) < 0 \). The final line follows from (88)-(90), the signs of the expressions, and the second derivative of \( \mathcal{L}(q) \). As \( \mathcal{L}(q) \) is twice continuously differentiable and its second derivative is negative for all \( q \in [0, q^*] \), then by Theorem M.C.2 from Mas-Colell, Whinston, and Green (1995), \( \mathcal{L}(q) \) is concave. \( \square \)

Proof of Lemma 4.4 Note that \( c''(q)[u'(q) - c'(q)]/u(q) \geq 0 \) for \( q \in [0, q^*] \). Therefore,

\[
u''(q) - c''(q) < 2c''(q)\frac{u''(q)c'(q) - c''(q)u'(q)}{[c'(q)]^3} - \frac{c''(q)[u'(q)]^2}{c(q)[c'(q)]^4} \leq 2c''(q)\frac{u''(q)c'(q) - c''(q)u'(q)}{[c'(q)]^3}.
\]
\( \square \)
Proof of Lemma 4.5

\[ u''(q) - c''(q) < 2c''(q) \frac{u''(q)c'(q) - c''(q)u'(q)}{[c'(q)]^3} \]

\[ \Rightarrow u''(q)c'(q) - c''(q)u'(q) < 2c''(q) \frac{u''(q)c'(q) - c''(q)u'(q)}{[c'(q)]^2} \]

\[ \Rightarrow u''(q)c'(q) - c''(q)u'(q) < 2c''(q) \frac{u''(q)c'(q) - c''(q)u'(q)}{[c'(q)]^2} \]

\[ \Rightarrow [u''(q)c'(q) - c''(q)u'(q)] \frac{[c'(q)]^2 \theta c(q) + (1 - \theta) u(q)}{u'(q)^4} < \]

\[ 2c''(q) \frac{u''(q)c'(q) - c''(q)u'(q)}{u'(q)^4} - \]

\[ c''(q) \frac{u'(q) - c'(q)}{c'(q)} - 2 \frac{[u''(q)c'(q) - c''(q)u'(q)]}{u'(q)} \]

\[ \Rightarrow c''(q) \frac{u'(q) - c'(q)}{c'(q)} + 2 \frac{[u''(q)c'(q) - c''(q)u'(q)]}{u'(q)} + \]

\[ [\theta c(q) + (1 - \theta) u(q)] \frac{[u''(q)c'(q) - c''(q)u'(q)]}{u'(q)^4} \]

\[ < 0 \]

\[ \Rightarrow \frac{d^2}{dq^2} [z(q) \mathcal{L}(q)] < 0. \]

The second line follows using the same reasoning as in the proof of Lemma 4.2. The third line is true because it is adding a positive term (subtracting a negative) to the right-hand
side. Note that in the penultimate step, the last expression on the left-hand side is negative. The final line follows from (88)-(90) and the fact the second derivative of \( z(q)\mathcal{L}(q) \) is less than or equal to the left-hand side of the previous line. As \( z(q)\mathcal{L}(q) \) is twice continuously differentiable and its second derivative is negative for all \( q \in [0,q^*] \), then by Theorem M.C.2 from Mas-Colell, Whinston, and Green (1995), \( z(q)\mathcal{L}(q) \) is concave.

Proof of Proposition 4.6
(i) Taking the corner solution to the buyer’s choice of assets, \((12)\), and setting \( d = 0 \) gives \(-r + \sigma \theta / (1 - \theta) < 0\). Therefore, \(-r + \sigma \theta / (1 - \theta) \geq 0\) is the region where the corner solution is not attained.
(ii) For any \( \theta < 1 \) and \( \sigma \in [0,1] \), there exists an \( r > \sigma \theta / (1 - \theta) \). From (i), this implies fiat is infeasible. As \( r \to \infty \), fiat is infeasible for all \((\sigma, \theta) \in [0,1] \times [0,1]\).
(iii) If \( r = 0 \), given that \( \sigma \in [0,1] \) and \( \theta \in [0,1] \), \( \sigma \theta / (1 - \theta) \geq 0 \) for all \((\sigma, \theta) \) pairs. From (i) this implies fiat is feasible for all \((\sigma, \theta) \) pairs.

Proof. Proof of Proposition 4.8
(i) Note that the sovereign’s problem does not change when \( r \) increases. This implies the sovereign will choose \((\phi, \kappa, A)\) such that the same \( q \) is chosen by buyers. To simplify matters, \( A \) is normalized to 1. Note that the demand for assets can be rewritten as \( \phi = z(q)/A - \kappa \). Also note that the pricing equation can be rewritten as \( \phi = \kappa \{(1 + r)/[r - \sigma \theta \mathcal{L}(q)] - 1\} \). In equilibrium, \( z(q)/A \), when \( A \) is normalized, will not change. This implies \( \kappa \{(1 + r)/[r - \sigma \theta \mathcal{L}(q)]\} \) must not change. When \( r \) increases, \( (1+r)/[r - \sigma \theta \mathcal{L}(q)] \) decreases. Therefore, \( \kappa \) must increase. This also implies \( \phi \) decreases.
(ii) The proof is similar to (i). The sovereign’s problem does not change when \( \sigma \) increases, so the sovereign will choose \((\phi, \kappa)\) such that the same \( q \) is chosen by buyers. Combining the pricing equation and the demand for assets implies \( \kappa \) must decrease, since \( (1 + r)/[r - \sigma \theta \mathcal{L}(q)] \) increases. This further implies \( \phi \) increases.
Proof. Proof of Proposition 5.2 Note that \( q \in [0, q^*] \subset \mathbb{R} \) and \( \alpha \in [0, 1] \subset \mathbb{R} \). Define \( f(q, \alpha) = (1 - \alpha)z(q)L(q) + \alpha[u(q) - c(q)]. \) Note that
\[
\frac{\partial^2}{\partial q \partial \alpha} f(q, \alpha) = -z(q)L'(q) \geq 0.
\]
This implies \( f(q, \alpha) \) has increasing differences in \( (q, \alpha) \). Then, by Corollary 2.6.1 from Topkis (1998), \( f(q, \alpha) \) is supermodular. These results satisfy the conditions for Theorem 2.8.1 from Topkis (1998). Therefore, \( \arg \max_{q \in [0, q^*], \alpha \in [0, 1]} f(q, \alpha) \) is increasing in \( \alpha \) over \( \alpha \in [0, 1] \).

Following similar reasoning as in the proof of Proposition 4.8, the demand for assets can be written as \( \phi = z(q)/A - \kappa \). The assets, \( A \) are normalized to one. From the pricing equation, \( \phi = \kappa \{(1 + r)/(r - \sigma \theta L(q)) - 1\} \). As \( \alpha \) increases, equilibrium \( q \) increases. This implies \( \kappa (1 + r)/(r - \sigma \theta L(q)) \) must increase. As \( q \) increases, \( L(q) \) decreases, therefore for the entire expression to increase, \( \kappa \) must increase. \( \square \)